Bandwidth ceiling of perfectly absorbing exceptional points: Lean 4 blueprint

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Chapter 1

Setting

We consider passive absorbers described by N resonant modes coupled to external channels, with intrinsic loss $L > 0$ (the total intrinsic amplitude-decay rate of the modes). Frequencies are detunings in the same units as L, so bandwidths are quoted as B/L. The *90% bandwidth* B is the width of the connected frequency interval that contains a coherent-perfect-absorption (CPA) frequency and on which the best input state absorbs at least 90%, i.e. $|r(\omega)|^2 \le 1/10$ for a single channel. An exceptional point (EP) of full order N is an N-mode absorber whose N CPA zeros all coalesce at one real frequency.

The main claims are:

- every perfectly absorbing full-order EP has $B \le 2\sqrt{2/3}\,L \approx 1.633\,L$, for every order $N \ge 2$, and the value is attained at $N = 2$;
- this is 29.9% of the Bode–Fano ceiling $4\pi L/\ln 10 \approx 5.458\,L$;
- splitting the zeros already beats every perfectly absorbing EP: a two-mode dimer reaches $(4/3)\sqrt{1+\sqrt{10}}\,L \approx 2.720\,L$.

Green nodes in the dependency graph are checked by Lean 4 with Mathlib; the remaining nodes are proved on paper (Supplemental Material of the Letter) but not yet formalized.

Chapter 2

The second-order EP attains $2\sqrt{2/3}\,L$

Definition 2.1. For $L > 0$ let $P(\omega) = \omega^2 + 2iL\omega - 2L^2$ and $r(\omega) = \omega^2/P(\omega)$, the reflection of a second-order EP with its double CPA zero at $\omega = 0$.

Lemma 2.2. *$P(\omega) = (\omega - (L - iL))(\omega - (-L - iL))$: the poles are $\pm L - iL$.*

Proof. Expand and use $i^2 = -1$. □

Lemma 2.3. *The trace identity $L = -\frac{1}{2}\sum_j (\operatorname{Im} p_j + \operatorname{Im} z_j)$ holds with poles $\pm L - iL$ and the double zero at 0, so this absorber has intrinsic loss exactly L.*

Proof. Direct computation. □

Lemma 2.4. *For real ω, $|P(\omega)|^2 = \omega^4 + 4L^4$.*

Proof. $|P|^2 = (\omega^2 - 2L^2)^2 + (2L\omega)^2$. □

Lemma 2.5. $|r(\omega)|^2 = \omega^4/(\omega^4 + 4L^4)$.

Proof. Divide. □

Theorem 2.6. $\{\omega \in \mathbb{R} : |r(\omega)|^2 \le 1/10\} = [-\sqrt{2/3}\,L,\ \sqrt{2/3}\,L]$. *Hence the 90% band of this second-order EP is exactly $2\sqrt{2/3}\,L$.*

Proof. $|r|^2 \le 1/10 \iff 9\omega^4 \le 4L^4 \iff \omega^2 \le \frac{2}{3}L^2$. □

Lemma 2.7. $1.6329 < 2\sqrt{2/3} < 1.6330$.

Proof. Square both sides. □

Chapter 3

Share of the Bode–Fano ceiling

Theorem 3.1. *(Bode–Fano ceiling.) Every passive absorber with a rational scattering matrix, lossless background and intrinsic loss L has $B \le 4\pi L/\ln 10$.*

Proof. Rank-one reduction, $\sigma_{\min}(S) \ge |\det S|$ and Jensen's formula give $\int_{\mathbb{R}} \ln|1/\det S|\, d\omega \le 2\pi L$; a 90% band contributes at least $\frac{1}{2}\ln 10$ per unit frequency. Not yet formalized. □

Lemma 3.2. $0.2985 < 2\sqrt{2/3}\,/\,(4\pi/\ln 10) < 0.2995$*, i.e. the EP ceiling is* 29.9% *of the Bode–Fano ceiling.*

Proof. Rigorous bounds on π, $\ln 2$ and $\ln(5/4)$. □

Chapter 4

Split zeros: the dimer reaches $2.720\,L$

Definition 4.1. With $L = 1$ and coupling $g \geq 1$, the dimer reflection is $r_g(\omega) = (\omega^2 + 1 - g^2)/(\omega^2 + 2i\omega - 1 - g^2)$, with real CPA zeros $\pm\sqrt{g^2 - 1}$.

Lemma 4.2. *With $u = \omega^2 - (g^2 - 1)$, $|r_g(\omega)|^2 = u^2/(u^2 + 4g^2)$.*

Proof. Expand the modulus of the denominator. □

Theorem 4.3. *For every $g \geq 1$, every order-connected set $I \subseteq \{\omega : |r_g(\omega)|^2 \leq 1/10\}$ containing the CPA zero $\sqrt{g^2 - 1}$ has width at most $(4/3)\sqrt{1 + \sqrt{10}}$.*

Proof. $|r_g|^2 \leq 1/10 \iff |u| \leq 2g/3$. For $g \leq (1+\sqrt{10})/3$ the band is $[-\sqrt{g^2 - 1 + 2g/3}, \sqrt{g^2 - 1 + 2g/3}]$, increasing in g; beyond, it splits into two pieces, each narrower. □

Theorem 4.4. *At $g = (1 + \sqrt{10})/3$ the interval $[-\frac{2}{3}\sqrt{1 + \sqrt{10}},\ \frac{2}{3}\sqrt{1 + \sqrt{10}}]$ lies in the 90% region and contains the CPA zero, so the bound of Theorem 4.3 is attained.*

Proof. At this g, $g^2 - 1 = 2g/3$, so the band edges are $\pm\sqrt{4g/3}$. □

Lemma 4.5. $2.720 < (4/3)\sqrt{1 + \sqrt{10}} < 2.7203$.

Proof. Square twice. □

Chapter 5

The EP envelope and the one-sided reach

Definition 5.1. $J_\theta = \begin{pmatrix} 1 & e^{i\theta} \\ -e^{-i\theta} & -1 \end{pmatrix}$ and the elementary transfer step $M_j(\omega) = I + \frac{2i\gamma_j}{\omega} J_{\theta_j}$, with loss weights $\gamma_j \geq 0$, $\sum_j \gamma_j = L$ and phases θ_j. In the canonical form of a full-order EP, $P(\omega)/\omega^N = \big(\prod_j M_j(\omega)\big)_{11}$.

Theorem 5.2. *(Canonical form.) Every perfectly absorbing full-order EP, with any number of ports and any loss matrix, reduces to the canonical form of Definition 5.1 without changing L.*

Proof. Rank-one reduction by Darlington synthesis, a unitary Schur transformation making the zero operator strictly upper triangular, and a diagonal gauge. Not yet formalized. □

Lemma 5.3. *For real $\omega \neq 0$, $\big|\big(\prod_j M_j(\omega)\big)_{11}\big| \leq \cosh \sum_j \operatorname{asinh}(2\gamma_j/|\omega|)$.*

Proof. Each M_j has diagonal entries of modulus $\cosh \rho_j$ and off-diagonal entries of modulus $\sinh \rho_j$, $\rho_j = \operatorname{asinh}(2\gamma_j/|\omega|)$; entry bounds of this form add under multiplication by $\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$. □

Lemma 5.4. $\sum_j \operatorname{asinh}(2\gamma_j/|\omega|) \leq N \operatorname{asinh}\big(2L/(N|\omega|)\big)$.

Proof. asinh is concave on $[0,\infty)$ (its derivative $(1+x^2)^{-1/2}$ is decreasing); apply Jensen's inequality with equal weights. □

Theorem 5.5. *If $|r(\omega)|^2 \leq 1/10$, i.e. $\big|\big(\prod_j M_j\big)_{11}\big| \geq \sqrt{10} = \cosh(\operatorname{asinh} 3)$, then $|\omega| \leq X_N = 2L/\big(N \sinh(\operatorname{asinh} 3/N)\big)$.*

Proof. Combine the two lemmas and invert cosh and sinh, both increasing on $[0,\infty)$. □

Lemma 5.6. *For every $N \geq 1$, $X_N < 2L/\operatorname{asinh} 3 \approx 1.100\, L$.*

Proof. $N \sinh(\kappa/N) > \kappa$ because $\sinh x > x$ for $x > 0$. □

Corollary 5.7. *Every perfectly absorbing full-order EP of any order has $B \leq 2X_N < 4L/\operatorname{asinh} 3 \approx 2.200\, L$.*

Proof. Theorem 5.5 applied on each side of the EP, after the reduction of Theorem 5.2. □

Chapter 6

The sharp ceiling for every order

Theorem 6.1. *For every $N \geq 2$, $\sup B_{\mathrm{EP}} = 2\sqrt{2/3}\,L \approx 1.633\,L$, attained at $N = 2$.*

Proof. Attainment is Theorem 2.6. The upper bound uses a Schwarz-chain representation of Hurwitz polynomials, convexity along adjacent chain couplings, an entropy dual for each multiplicity of the coalesced zero, a second-order expansion around the dimer and 984 cells checked in 160-bit interval arithmetic (Supplemental Material, Secs. S5, S6 and S16). Not yet formalized. □